

SIMILARITY OF OPERATORS IN THE BERGMAN SPACE SETTING

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ABSTRACT. We give a necessary and sufficient condition for an n -hypercontraction to be similar to the backward shift operator in a weighted Bergman space. This characterization serves as a generalization of the description given in the Hardy space setting, where the geometry of the eigenvector bundles of the operators is used.

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NOTATION

$:=$	equal by definition;
\mathbb{C}	the complex plane;
\mathbb{D}	the unit disk, $\mathbb{D} := \{z \in \mathbb{C} : z < 1\}$;
\mathbb{T}	the unit circle, $\mathbb{T} := \partial\mathbb{D} = \{z \in \mathbb{C} : z = 1\}$;

2000 *Mathematics Subject Classification.* Primary 47A99, Secondary 47B32, 30D55, 53C55.

Key words and phrases. Cowen-Douglas class, n -hypercontraction, similarity, weighted Bergman space, eigenvector bundle, backward shift, reproducing kernel.

The work of R. G. Douglas was partially supported by a grant from the National Science Foundation. The work of H. Kwon was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science, and Technology (2011-0026989) and by the T. J. Park Postdoctoral Fellowship. The work of S. Treil was supported by the National Science Foundation under Grant DMS-0800876.

$\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$	∂ and $\bar{\partial}$ derivatives: $\frac{\partial}{\partial z} := (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})/2$, $\frac{\partial}{\partial \bar{z}} := (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})/2$;
Δ	normalized Laplacian, $\Delta := \bar{\partial}\partial = \partial\bar{\partial} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$;
\mathfrak{S}_2	Hilbert-Schmidt class of operators;
$\ \cdot\ , \ \cdot\ $	norm: since we are dealing with matrix- and operator-valued functions, we will use the symbol $\ \cdot\ $ (usually with a subscript) for the norm in a function space, while $\ \cdot\ $ is used for the norm in the underlying vector (operator) space. Thus, for a vector-valued function f the symbol $\ f\ _2$ denotes its L^2 -norm, but the symbol $\ f\ $ stands for the scalar-valued function whose value at a point z is the norm of the vector $f(z)$;
H^∞	the space of all functions bounded and analytic in \mathbb{D} ;
$L_{E_* \rightarrow E}^\infty$	class of bounded functions on the unit circle \mathbb{T} whose values are bounded operators from a Hilbert space E_* to another one E (the spaces E and E_* are not supposed to be related in any way);
$H_{E_* \rightarrow E}^\infty$	operator Hardy class of bounded analytic functions whose values are bounded operators from E_* to E :
$\ F\ _\infty := \sup_{z \in \mathbb{D}} \ F(z)\ = \operatorname{esssup}_{\xi \in \mathbb{T}} \ F(\xi)\ ;$	
T_Φ	Toeplitz operator with symbol Φ .

All Hilbert spaces are assumed to be separable. We also assume that in a Hilbert space, an orthonormal basis is fixed so that any operator $A : E \rightarrow E_*$ can be identified with its matrix. Thus, besides the usual involution $A \mapsto A^*$ (A^* is the adjoint of A), we have two more: $A \mapsto A^T$ (transpose of the matrix) and $A \mapsto \overline{A}$ (complex conjugation of the matrix), so $A^* = (\overline{A})^T = \overline{A^T}$. Although everything in the paper can be presented in an invariant, “coordinate-free” form, the use of the transposition and complex conjugation makes the notation simpler and more transparent.

0. INTRODUCTION

We consider the question of when operators with a complete analytic family of eigenvectors are similar. Recall that operators T_1 and T_2 are said to be *similar* if there exists a bounded, invertible operator A satisfying the intertwining relation $AT_1 = T_2A$.

The problem of determining when two such operators are unitarily equivalent goes back to the 1970’s when the Cowen-Douglas class was introduced in [4]. It is proven there that unitary equivalence has to do with the curvatures of the eigenvector bundles of the operators and the partial derivatives of them up to a certain order matching up. Unlike the unitary equivalence case, however, the similarity problem posed a more complicated situation

(only some necessary conditions are listed in [4]) and no such criterion was obtained.

By adding the assumption that the operators in consideration be contractive ($\|T\| \leq 1$), the authors in [8] dealt with a special case of the problem; they gave a description of operators with a complete analytic family of eigenvectors that are similar to S^* , the backward shift operator on the Hardy space H^2 (both scalar- and vector-valued) of the unit disk \mathbb{D} . The backward shift S^* is defined to be the adjoint of the forward shift S ,

$$Sf(z) = zf(z),$$

for $f \in H^2$, and similarity is shown to be equivalent to the existence of a bounded (subharmonic) solution φ defined on \mathbb{D} to the Poisson equation

$$\Delta\varphi = g,$$

where g is a function related to the curvatures of the eigenvector bundles of the operators.

One can ask whether the above characterization also holds for the backward shift operators B_α^* defined on the weighted Bergman spaces A_α^2 (again, both scalar- and vector-valued) of \mathbb{D} . If we let P_α denote the Bergman projection and let T_Φ be the Toeplitz operator with symbol Φ given by

$$T_\Phi f = P_\alpha(\Phi f),$$

then it is easily seen that our backward shifts can be represented for $f \in A_\alpha^2$ as

$$B_\alpha^* f(z) = P_\alpha(\bar{z}f(z)) = T_{\bar{z}}f(z),$$

just like in the Hardy space case where the Bergman projections are replaced by the Szegő projection. We show in this paper that the function-theoretic proof provided in [8] for S^* on H^2 can be applied to B_α^* on A_α^2 , giving a generalization of the results there. Finally we mention the recent paper [5], where the authors use a Hilbert module approach to prove that the similarity to the backward shift operator on certain reproducing kernel Hilbert spaces can be reduced to the similarity to S^* on H^2 .

1. PRELIMINARIES

Let n be a positive integer. Following the notation of [2], we denote by \mathcal{M}_n the Hilbert space of analytic functions on the unit disk \mathbb{D} satisfying

$$\|f\|_n^2 := \sum_{i=0}^{\infty} |\hat{f}(i)|^2 \frac{1}{\binom{n+i-1}{i}} < \infty,$$

for $\mathcal{M}_n \ni f = \sum_{i=0}^{\infty} \hat{f}(i)z^i$. Note that \mathcal{M}_n corresponds to the Hardy space H^2 for $n = 1$, and for each positive integer $n \geq 2$, to the weighted Bergman space A_{n-2}^2 defined by

$$A_{n-2}^2 = \{f \in \text{Hol}(\mathbb{D}) : (n-1) \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{n-2} dA(z) < \infty\},$$

for dA the normalized area measure on \mathbb{D} . We can define the vector-valued spaces $\mathcal{M}_{n,E}$ taking values in a separable Hilbert space E in a similar way.

On the space $\mathcal{M}_{n,E}$ are the forward shift operator $S_{n,E}$, $S_{n,E}f(z) = zf(z)$ and the backward shift operator $S_{n,E}^*$, its adjoint. Since \mathcal{M}_n is a reproducing kernel Hilbert space with reproducing kernel $k_\lambda^n := (1 - \bar{\lambda}z)^{-n}$, $\lambda \in \mathbb{D}$, the eigenvectors of $S_{n,E}^*$ corresponding to the eigenvalue λ is $k_\lambda^n e$ for $e \in E$.

We now come to the definition of an n -hypercontraction introduced in [1] and [2]. Let H be a Hilbert space. An operator $T \in \mathcal{L}(H)$ is called an n -hypercontraction if

$$\sum_{i=0}^k (-1)^i \binom{k}{i} T^{*i} T^i \geq 0,$$

for all $1 \leq k \leq n$. Note that the 1-hypercontraction case corresponds to the definition of the usual contraction.

Lastly, we recall the definition of a Carleson measure. Let

$$Q(I) := \{z \in \mathbb{T} : \frac{z}{|z|} \in I, 1 - |z| \leq |I|\},$$

for $I \subseteq \mathbb{T}$, an arc of length $|I|$. A complex measure μ in the closed unit disk is called a *Carleson measure* if for some constant C ,

$$|\mu|Q(I) \leq C|I|,$$

where $|\mu|$ denotes the variation of μ [9].

2. MAIN RESULTS

Let n be a positive integer and H a Hilbert space. We assume the following for the operator $T \in \mathcal{L}(H)$ that we consider:

- (1) T is an n -hypercontraction;
- (2) $\text{span}\{\ker(T - \lambda) : \lambda \in \mathbb{D}\} = H$; and
- (3) $\ker(T - \lambda)$ depend analytically on the spectral parameter $\lambda \in \mathbb{D}$.

Assumption (3) says that for each $\lambda \in \mathbb{D}$, a neighborhood U_λ of λ and an operator-valued analytic function F_λ defined on U_λ that is left-invertible in L^∞ satisfying

$$\text{ran } F_\lambda(w) = \ker(T - w),$$

for all $w \in U_\lambda$ exist. Therefore, the disjoint union $\coprod_{\lambda \in \mathbb{D}} \ker(T - \lambda) = \{(\lambda, v_\lambda) : \lambda \in \mathbb{D}, v_\lambda \in \ker(T - \lambda)\}$ is a hermitian, holomorphic vector bundle over \mathbb{D} with the metric inherited from H and the natural projection π , $\pi(\lambda, v_\lambda) = \lambda$. Note that assumption (3) then implies that $\dim \ker(T - \lambda)$ is constant for all $\lambda \in \mathbb{D}$. According to [4], the operators that belong to the Cowen-Douglas class $B_m(\mathbb{D})$, or more generally those with a certain Fredholm condition, for instance, satisfy assumption (3).

We next mention that a *bundle map* is a holomorphic map between two holomorphic vector bundles over \mathbb{D} that linearly maps each fiber $\pi^{-1}(\lambda)$ of one bundle to the corresponding fiber of the other bundle.

Now we state the main results of the paper:

Theorem 2.1. *Let $T \in \mathcal{L}(H)$ satisfy the above 3 assumptions with $\dim \ker(T - \lambda) = m < \infty$ for every $\lambda \in \mathbb{D}$. Denote by $\Pi : \mathbb{D} \rightarrow \mathcal{L}(H)$ the projection-valued function that assigns to each $\lambda \in \mathbb{D}$, the orthogonal projection onto $\ker(T - \lambda)$. The following statements are equivalent:*

- (1) *T is similar to the backward shift operator S_{n, \mathbb{C}^m}^* on $\mathcal{M}_{n, \mathbb{C}^m}$ via an invertible operator $A : \mathcal{M}_{n, \mathbb{C}^m} \rightarrow H$;*
- (2) *There exists a holomorphic bundle map bijection Ψ from the eigenvector bundle of S_{n, \mathbb{C}^m}^* to that of T such that for some constant $c > 0$,*

$$\frac{1}{c} \|v_\lambda\|_{\mathcal{M}_{n, \mathbb{C}^m}} \leq \|\Psi(v_\lambda)\|_H \leq c \|v_\lambda\|_{\mathcal{M}_{n, \mathbb{C}^m}},$$

for all $v_\lambda \in \ker(S_{n, \mathbb{C}^m}^ - \lambda)$ and for all $\lambda \in \mathbb{D}$;*

- (3) *There exists a bounded solution φ defined on \mathbb{D} to the Poisson equation*

$$\Delta \varphi(z) = \left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1 - |z|^2)^2}.$$

Corollary 2.2. *A contraction T that satisfies assumptions (2), (3), and*

$$\sum_{i=0}^n (-1)^i \binom{n}{i} T^{*i} T^i \geq 0,$$

enjoys the similarity characterization given in Theorem 2.1.

Corollary 2.3. *A subnormal contraction that satisfies assumptions (2) and (3) enjoys the similarity characterization given in Theorem 2.1.*

Remark 2.4. Note that the function Π is C^∞ and even real analytic in the operator norm topology, so it does make sense to consider $\frac{\partial \Pi(z)}{\partial z}$.

Remark 2.5. Since $\left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1 - |z|^2)^2} \geq 0$ (see Section 3), φ is actually subharmonic.

Remark 2.6. For $m = 1$, $-\left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2$ and $-\frac{n}{(1 - |z|^2)^2}$ represent the curvatures of the eigenvector bundles of T and of $S_{n, \mathbb{C}}^*$, respectively ([4], [7]).

Remark 2.7. The existence of a bounded subharmonic function φ defined on \mathbb{D} satisfying

$$\Delta \varphi(z) \geq \left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1 - |z|^2)^2}$$

is equivalent to the uniform boundedness of the Green potential

$$\mathcal{G}(\lambda) := \frac{2}{\pi} \iint_{\mathbb{D}} \log \left| \frac{z - \lambda}{1 - \bar{\lambda}z} \right| \left(\left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1 - |z|^2)^2} \right) dx dy$$

inside the unit disk \mathbb{D} .

In order to prove Theorem 2.1, we first need to obtain a tensor product structure for the operator T . Then since the equivalence of statements (1) and (2) of Theorem 2.1 is obvious, and (3) follows from the two statements (4) The measure

$$\left(\left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1-|z|^2)^2} \right) (1-|z|) dx dy$$

is Carleson; and

(5) We have the estimate

$$\left(\left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1-|z|^2)^2} \right)^{\frac{1}{2}} \leq \frac{C}{1-|z|},$$

it suffices to show that (2) implies both (4) and (5) (Section 4) and that (3) implies (1) (Section 5).

3. TENSOR STRUCTURE OF THE EIGENVECTOR BUNDLE

3.1. Structure of the eigenvector bundle of T . The following theorem by J. Agler ([2]) proven through the Rovnyak-de Branges construction is the first step to obtaining a tensor product representation of the eigenvector bundle of T . The reader is advised to consult [1] also for an alternative proof of the theorem based on complete positivity:

Theorem 3.1. *Let $T \in \mathcal{L}(H)$. There exists a Hilbert space E and an $S_{n,E}^*$ -invariant subspace $\mathcal{N} \subseteq \mathcal{M}_{n,E}$ such that T is unitarily equivalent to $S_{n,E}^*|_{\mathcal{N}}$ if and only if T is an n -hypercontraction with $\lim_k \|T^k h\| = 0$ for all $h \in H$.*

Let us first observe that $\lim_k \|T^k h\| = 0$ for $h \in H$ that is a linear combination of the eigenvectors of T . According to assumption (2), these linear combinations form a dense subspace of H . Moreover, since an n -hypercontraction is automatically a contraction, we have $\|T^k\| \leq 1$. We can thus employ a standard argument to show that $\lim_k \|T^k h\| = 0$ for all $h \in H$.

Hence, the eigenspaces of $T = S_{n,E}^*|_{\mathcal{N}}$ are given by

$$\ker(T - \lambda) = \{k_{\bar{\lambda}}^n e : e \in \mathcal{N}(\lambda)\},$$

where $k_{\bar{\lambda}}^n = (1 - \bar{\lambda}z)^{-n}$, $\lambda \in \mathbb{D}$, is the reproducing kernel for \mathcal{M}_n and $\mathcal{N}(\lambda) := \{e \in E; k_{\bar{\lambda}}^n e \in \mathcal{N}\}$. Note that by assumption (3), the subspaces $\mathcal{N}(\lambda)$ also depend analytically on the spectral parameter λ , i.e., the family of subspaces $\mathcal{N}(\lambda)$ is a holomorphic vector bundle over \mathbb{D} .

Now, since the vector-valued Hilbert space $\mathcal{M}_{n,E}$ can be identified with $\mathcal{M}_n \otimes E$, the tensor product of the Hilbert spaces \mathcal{M}_n and E , the eigenvector bundle of T takes on the form

$$\ker(T - \lambda) = \text{span}\{k_{\bar{\lambda}}^n\} \otimes \mathcal{N}(\lambda).$$

3.2. Calculation involving the eigenvector bundle of T . Recall that $\Pi(\lambda)$ stands for the orthogonal projection onto $\ker(T - \lambda)$. Using the tensor structure given above, we can express $\Pi(\lambda)$ as

$$(3.1) \quad \Pi(\lambda) = \Pi_1(\lambda) \otimes \Pi_2(\lambda),$$

where $\Pi_1(\lambda)$ is the orthogonal projection from the space \mathcal{M}_n onto $\text{span}\{k_\lambda^n\}$, and $\Pi_2(\lambda)$ is the orthogonal projection from E onto $\mathcal{N}(\lambda)$. We remark that $\text{rank } \Pi(\lambda) = \text{rank } \Pi_2(\lambda) = m$.

Lemma 3.2. *For $\lambda \in \mathbb{D}$, let $\Gamma(\lambda)$ be orthogonal projections onto an analytic family of subspaces (holomorphic vector bundle). Then the identities*

$$\Gamma(z) \frac{\partial \Gamma(z)}{\partial z} = 0$$

and

$$(I - \Gamma(z)) \frac{\partial \Gamma(z)}{\partial z} \Gamma(z) = \frac{\partial \Gamma(z)}{\partial z}$$

hold.

Proof of Lemma 3.2. Since the family of subspaces is a holomorphic vector bundle, it can be locally expressed as $\text{ran } F(\lambda)$, where F is an analytic, left-invertible operator-valued function. Thus, $\Gamma = F(F^*F)^{-1}F^*$. We obtain through direct computation that

$$\frac{\partial \Gamma(z)}{\partial z} = (I - \Gamma(z))F'(z)(F(z)^*F(z))^{-1}F(z)^*.$$

Since $\Gamma(z)$ is a projection, we immediately arrive at the first identity. For the second one, we note that $\Gamma(z)F(z) = F(z)$ implies $\frac{\partial \Gamma(z)}{\partial z} \Gamma(z) = \frac{\partial \Gamma(z)}{\partial z}$. We then invoke the first identity. \square

Lemma 3.3. *The projection $\Pi_1(\lambda)$ satisfies the identity*

$$\left\| \frac{\partial \Pi_1(z)}{\partial z} \right\|_{\mathfrak{S}_2}^2 = n(1 - |z|^2)^{-2}.$$

Proof of Lemma 3.3. We first use the reproducing kernel property of $k_\lambda^n = 1/(1 - \bar{\lambda}z)^n$ to see that $\|k_\lambda^n\|_2^2 = \langle k_\lambda^n, k_\lambda^n \rangle = (1 - |\lambda|^2)^{-n}$. Thus

$$\Pi_1(\lambda)f = \|k_\lambda^n\|_2^{-2} \langle f, k_\lambda^n \rangle k_\lambda^n = (1 - |\lambda|^2)^n f(\bar{\lambda}) k_\lambda^n,$$

for $f \in M_n$. We next use the fact that $\frac{\partial f(\bar{\lambda})}{\partial \lambda} = 0$ and $\frac{\partial}{\partial \lambda} k_\lambda^n(z) = \frac{nz}{(1 - \lambda z)^{n+1}} =: \tilde{k}_\lambda^n(z)$ to get

$$(3.2) \quad \frac{\partial \Pi_1(\lambda)}{\partial \lambda} f = (1 - |\lambda|^2)^{n-1} f(\bar{\lambda}) \left(-n\bar{\lambda} k_\lambda^n + (1 - |\lambda|^2) \tilde{k}_\lambda^n \right).$$

Since $\langle f, \tilde{k}_\lambda^n \rangle = f'(\lambda)$ for $f \in M_n$,

$$\|\tilde{k}_\lambda^n\|_2^2 = \frac{n(1 + n|\lambda|^2)}{(1 - |\lambda|^2)^{n+2}} = \|\tilde{k}_\lambda^n\|_2^2.$$

Once again, the reproducing property of k_λ^n implies that

$$\langle \tilde{k}_\lambda^n, k_\lambda^n \rangle = \frac{n\bar{\lambda}}{(1 - |\lambda|^2)^{n+1}}.$$

Taking all these calculations into account, we conclude that

$$\| -n\bar{\lambda}k_\lambda^n + (1 - |\lambda|^2)\tilde{k}_\lambda^n \|_2^2 = n(1 - |\lambda|^2)^{-n}.$$

Thus,

$$\left| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right|^2 = n(1 - |\lambda|^2)^{-2},$$

and we note from (3.2) that

$$\text{rank} \frac{\partial \Pi_1(\lambda)}{\partial \lambda} = 1.$$

Therefore,

$$\left| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 = \left| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right|^2 = n(1 - |\lambda|^2)^{-2}.$$

□

Lemma 3.4. *The projection $\Pi(\lambda)$ satisfies the identity*

$$\begin{aligned} \left| \frac{\partial \Pi(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 &= m \left| \frac{\partial \Pi_1(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 + \left| \frac{\partial \Pi_2(z)}{\partial z} \right|_{\mathfrak{S}_2}^2 \\ &= \frac{mn}{(1 - |z|^2)^2} + \left| \frac{\partial \Pi_2(z)}{\partial z} \right|_{\mathfrak{S}_2}^2. \end{aligned}$$

Proof of Lemma 3.4. We apply the product rule to (3.1) to obtain

$$\frac{\partial \Pi(\lambda)}{\partial \lambda} = \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \otimes \Pi_2(\lambda) + \Pi_1(\lambda) \otimes \frac{\partial \Pi_2(\lambda)}{\partial \lambda} =: X + Y.$$

Since $\Pi_2(\lambda) \frac{\partial \Pi_2(\lambda)}{\partial \lambda} = 0$ by Lemma 3.2, $X^*Y = 0$. Therefore,

$$\|X + Y\|_{\mathfrak{S}_2}^2 = \text{tr } X^*X + \text{tr } Y^*Y + 2 \text{Re tr}(X^*Y) = \|X\|_{\mathfrak{S}_2}^2 + \|Y\|_{\mathfrak{S}_2}^2.$$

Using the fact that $\|A \otimes B\|_{\mathfrak{S}_2}^2 = \|A\|_{\mathfrak{S}_2}^2 \|B\|_{\mathfrak{S}_2}^2$ and that $\|P\|_{\mathfrak{S}_2}^2 = \text{rank } P$ for an orthogonal projection P , we get

$$\left| \frac{\partial \Pi(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 = m \left| \frac{\partial \Pi_1(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 + \left| \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2.$$

The result now follows from Lemma 3.3. □

4. PROOF OF “(2) IMPLIES (3)”

Let us mention again that statements (4) and (5) of Section 2 together imply statement (3) of Theorem 2.1. Moreover, since we have by Lemma 3.4

$$\left| \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 = \left| \frac{\partial \Pi(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1-|\lambda|^2)^2},$$

the quantity $\left| \frac{\partial \Pi(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2 - \frac{mn}{(1-|\lambda|^2)^2}$ in statements (4) and (5) can be replaced by $\left| \frac{\partial \Pi_2(\lambda)}{\partial \lambda} \right|_{\mathfrak{S}_2}^2$.

Assume that statement (2) of Theorem 2.1 holds to guarantee the existence of a holomorphic bundle map bijection Ψ with a certain property between the eigenvector bundles. Then for all $e \in \mathbb{C}^m$,

$$\Psi(k_\lambda^n e) = k_\lambda^n \cdot F(\lambda)e,$$

where F is some function in $H_{\mathbb{C}^m \rightarrow E}^\infty$ satisfying $\text{ran } F(\lambda) = \mathcal{N}(\lambda)$ and $c^{-1}I \leq F^*F \leq cI$. Thus it makes sense to consider $(F^*F)^{-1}$ and we can express the orthogonal projection $\Pi_2(\lambda)$ from E onto $\mathcal{N}(\lambda)$ in terms of F as

$$\Pi_2 = F(F^*F)^{-1}F^*.$$

Since $\frac{\partial \Pi_2(z)}{\partial z} = (I - \Pi_2(z))F'(z)(F(z)^*F(z))^{-1}F(z)^*$, we get

$$(4.1) \quad \left| \frac{\partial \Pi_2(z)}{\partial z} \right| \leq C \|F'(z)\|.$$

Lastly, we note that since F is a bounded analytic function taking values in a Hilbert space, the estimate

$$(4.2) \quad \|F'(z)\| \leq C/(1-|z|)$$

holds, and the measure

$$(4.3) \quad \|F'(z)\|^2(1-|z|)dxdy$$

is Carleson. The first estimate (4.2) is well-known for scalar-valued analytic functions, and one can pick $x^* = x^*(z)$, $\|x^*\| = 1$ in the dual space X^* such that $\langle F'(z), x^* \rangle = \|F'(z)\|$ to show that it holds for functions with values in a Banach space X . To see that the Carleson measure condition (4.3) holds, we use *Uchiyama's Lemma* which states that for a bounded subharmonic function u , the measure $\Delta u(z)(1-|z|)dxdy$ is Carleson. We apply this Lemma to the function $u(z) = \|F(z)\|^2$ and note that $\Delta \|F(z)\|^2 = \|F'(z)\|^2$. By (4.1), (4.2), and (4.3), we get the existence of a bounded subharmonic function φ on \mathbb{D} with

$$\Delta \varphi(z) \geq \left| \frac{\partial \Pi_2(z)}{\partial z} \right|_{\mathfrak{S}_2}^2.$$

To obtain equality, we note that the equation $\Delta u(z) = f(z)$ always has a solution, namely, the Green potential

$$\mathcal{G}_f(\lambda) := \frac{2}{\pi} \iint_{\mathbb{D}} \log \left| \frac{z - \lambda}{1 - \bar{\lambda}z} \right| f(z) dx dy.$$

But since

$$G_{\Delta\varphi} \leq G \left| \frac{\partial \Pi_2}{\partial z} \right|_{\mathfrak{S}_2}^2 \leq 0,$$

and $G_{\Delta\varphi}$ is bounded, the subharmonic solution $G \left| \frac{\partial \Pi_2}{\partial z} \right|_{\mathfrak{S}_2}^2$ to

$$\Delta u(z) = \left| \frac{\partial \Pi_2(z)}{\partial z} \right|_{\mathfrak{S}_2}^2$$

is bounded as well.

5. PROOF OF “(3) IMPLIES (1)”

The goal of this section is to prove the existence of a bounded, invertible operator $A : \mathcal{M}_{n, \mathbb{C}^m} \rightarrow \mathcal{N}$ such that $AS_{n, \mathbb{C}^m}^* = (S_{n, E}^* | \mathcal{N})A$. We first consider the following theorem that will let us get a bounded, analytic projection onto $\text{ran } \mathcal{N}(z)$ for $z \in \mathbb{D}$ [11].

Theorem 5.1. *Let $\Gamma : \mathbb{D} \rightarrow \mathcal{L}(H)$ be a \mathcal{C}^2 function whose values are orthogonal projections in H . Assume that Γ satisfies the identity $\Gamma(z) \frac{\partial \Gamma(z)}{\partial z} = 0$ for all $z \in \mathbb{D}$. Given a bounded, subharmonic function φ with*

$$\Delta\varphi(z) \geq \left| \frac{\partial \Gamma(z)}{\partial z} \right|^2 \quad \text{for all } z \in \mathbb{D},$$

there exists a bounded analytic projection onto $\Gamma(z)$, i.e., a function $\mathcal{P} \in H_{H \rightarrow H}^\infty$ such that $\mathcal{P}(z)$ is a projection onto $\text{ran } \Gamma(z)$ for all $z \in \mathbb{D}$.

We know from Lemma 3.2 that the function Π_2 whose values are orthogonal projections from E onto $\mathcal{N}(\lambda)$ satisfies the identity $\Pi_2(z) \frac{\partial \Pi_2(z)}{\partial z} = 0$ so that the above theorem is applicable. We thus get a bounded, analytic projection $\mathcal{P}(z)$ onto $\text{ran } \Pi_2(z) = \mathcal{N}(z)$, and consider the inner-outer factorization $\mathcal{P} = \mathcal{P}_i \mathcal{P}_o$ of \mathcal{P} , where $\mathcal{P}_i \in H_{E_* \rightarrow E}^\infty$ for some Hilbert space E_* , is an inner function and $\mathcal{P}_o \in H_{E \rightarrow E_*}^\infty$ is an outer function. We then define a function \mathcal{Q}_i on \mathbb{D} by

$$\mathcal{Q}_i(z) := \mathcal{P}_i(\bar{z}),$$

and form the anti-analytic Toeplitz operator $T_{\mathcal{Q}_i}$.

We claim that this bounded Toeplitz operator $T_{\mathcal{Q}_i}$ is an invertible operator that establishes similarity. To this end, we need to prove the following three statements:

- (1) $T_{\bar{z}} T_{\mathcal{Q}_i} = T_{\mathcal{Q}_i} T_{\bar{z}}$;
- (2) $T_{\mathcal{Q}_i}$ is left-invertible; and
- (3) $\text{ran } T_{\mathcal{Q}_i} = \mathcal{N}$.

We begin by recalling some well-known facts about Toeplitz operators on the vector-valued spaces \mathcal{M}_n . Let $F, G \in H_{E \rightarrow E_*}^\infty$:

$$(5.1) \quad T_{FG} = T_F T_G; \text{ and}$$

$$(5.2) \quad T_{F^*} k_\lambda^n e = k_\lambda^n F^*(\lambda) e \text{ for } e \in E_*.$$

Since $\mathcal{Q}_i^* \in H_{E \rightarrow E_*}^\infty$, statement (1) easily follows from (5.1). To prove (2), we consider the following Lemma.

Lemma 5.2. $\mathcal{P}_o(z) \mathcal{P}_i(z) \equiv I$ for all $z \in \mathbb{D}$.

Proof of Lemma 5.2. By (5.1), we have that

$$T_{\mathcal{P}_i} T_{\mathcal{P}_o} = T_{\mathcal{P}} = T_{\mathcal{P}^2} = T_{\mathcal{P}_i \mathcal{P}_o \mathcal{P}_i \mathcal{P}_o} = T_{\mathcal{P}_i} T_{\mathcal{P}_o \mathcal{P}_i} T_{\mathcal{P}_o}.$$

Since $T_{\mathcal{P}_o}$ has dense range and $\ker T_{\mathcal{P}_i} = \{0\}$, $T_{\mathcal{P}_o \mathcal{P}_i} = I$, so $\mathcal{P}_o \mathcal{P}_i \equiv I$ for all $z \in \mathbb{D}$. \square

We then note that since $\mathcal{Q}_o^* \in H_{E_* \rightarrow E}^\infty$, where $\mathcal{Q}_o(z) := \mathcal{P}_o(\bar{z})$, we can once again use (5.1) to conclude that

$$T_{\mathcal{Q}_o} T_{\mathcal{Q}_i} = T_{\mathcal{Q}_o \mathcal{Q}_i} = I.$$

It now remains to show statement (3). The inclusion $\mathcal{N}(\lambda) = \text{ran } \mathcal{P}(\lambda) \subset \text{ran } \mathcal{P}_i(\lambda)$ is obvious due to the factorization $\mathcal{P} = \mathcal{P}_i \mathcal{P}_o$. For the other inclusion, since $\text{ran } \mathcal{P}_o(\lambda)$ is dense in E_* for all $\lambda \in \mathbb{D}$, and $\mathcal{P}_i(\lambda) \text{ran } \mathcal{P}_o(\lambda) = \mathcal{N}(\lambda)$, $\text{ran } \mathcal{P}_i(\lambda) \subset \mathcal{N}(\lambda)$. Thus,

$$(5.3) \quad \text{ran } \mathcal{P}_i(\lambda) = \mathcal{N}(\lambda).$$

We next observe that by (5.2),

$$(5.4) \quad T_{\mathcal{Q}_i} k_\lambda^n e = k_\lambda^n \mathcal{Q}_i(\bar{\lambda}) e = k_\lambda^n \mathcal{P}_i(\lambda) e,$$

for all $e \in E_*$. Then (3) follows from (5.4), the fact that $\text{span}\{k_\lambda^n : \lambda \in \mathbb{D}\} = M_n$, and assumption (2) that $\text{span}\{\ker(T - \lambda) : \lambda \in \mathbb{D}\} = H$. \square

6. PROOF OF COROLLARIES

Now we prove the corollaries of Theorem 2.1 that appeared in Section 2. The statements used in these proofs are contained in [2].

Proof of Corollary 2.2. We have $\lim_k \|T^k h\| = 0$ for $h \in H$ that is a linear combination of the eigenvectors of T , which by assumption (2) is dense in H . If T is a contraction, then $\|T^k\| \leq 1$, so that $\lim_k \|T^k h\| = 0$ for all $h \in H$. Now we use the result that an operator $T \in \mathcal{L}(H)$ with

$$\sum_{i=0}^n (-1)^i \binom{n}{i} T^{*i} T^i \geq 0,$$

and such that $\lim_k \|T^k h\| = 0$ for all $h \in H$ is an n -hypercontraction. \square

Proof of Corollary 2.3. An operator T is an n -hypercontraction for every n if and only if $\|T\| \leq 1$ and T is subnormal [6]. \square

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